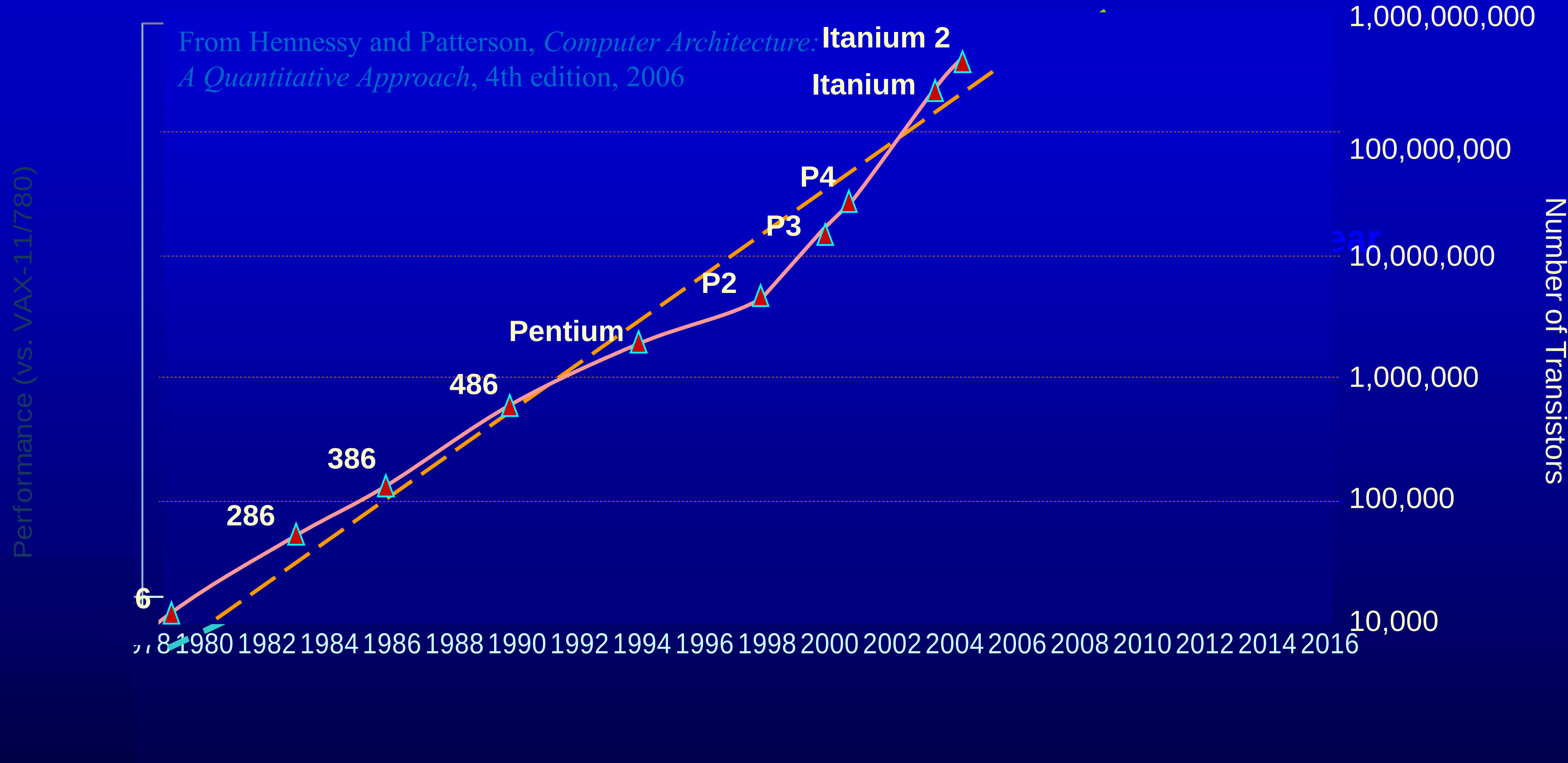
6.035

Parallelization

Outline

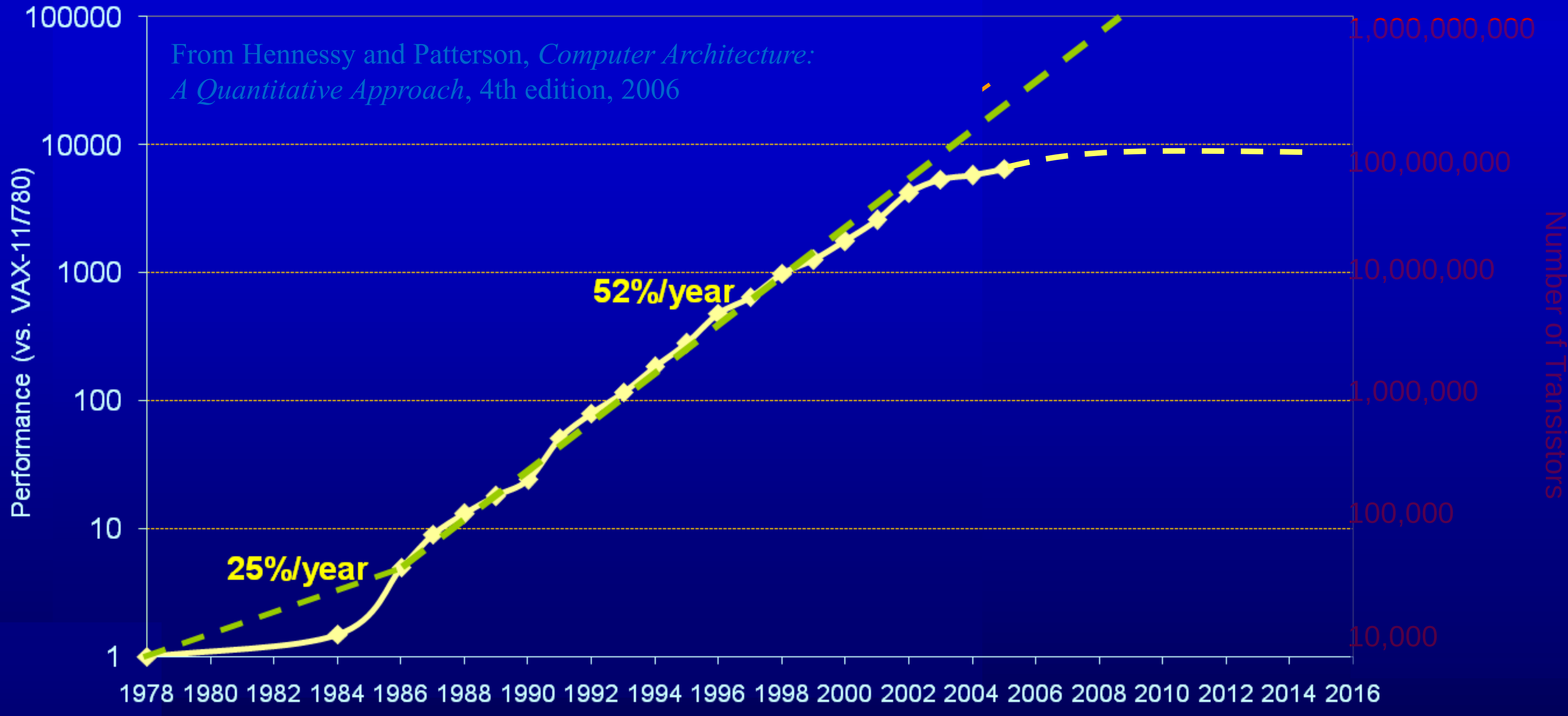
- **Why Parallelism**
- Parallel Execution
- Parallelizing Compilers
- Dependence Analysis
- Increasing Parallelization Opportunities

Moore's Law



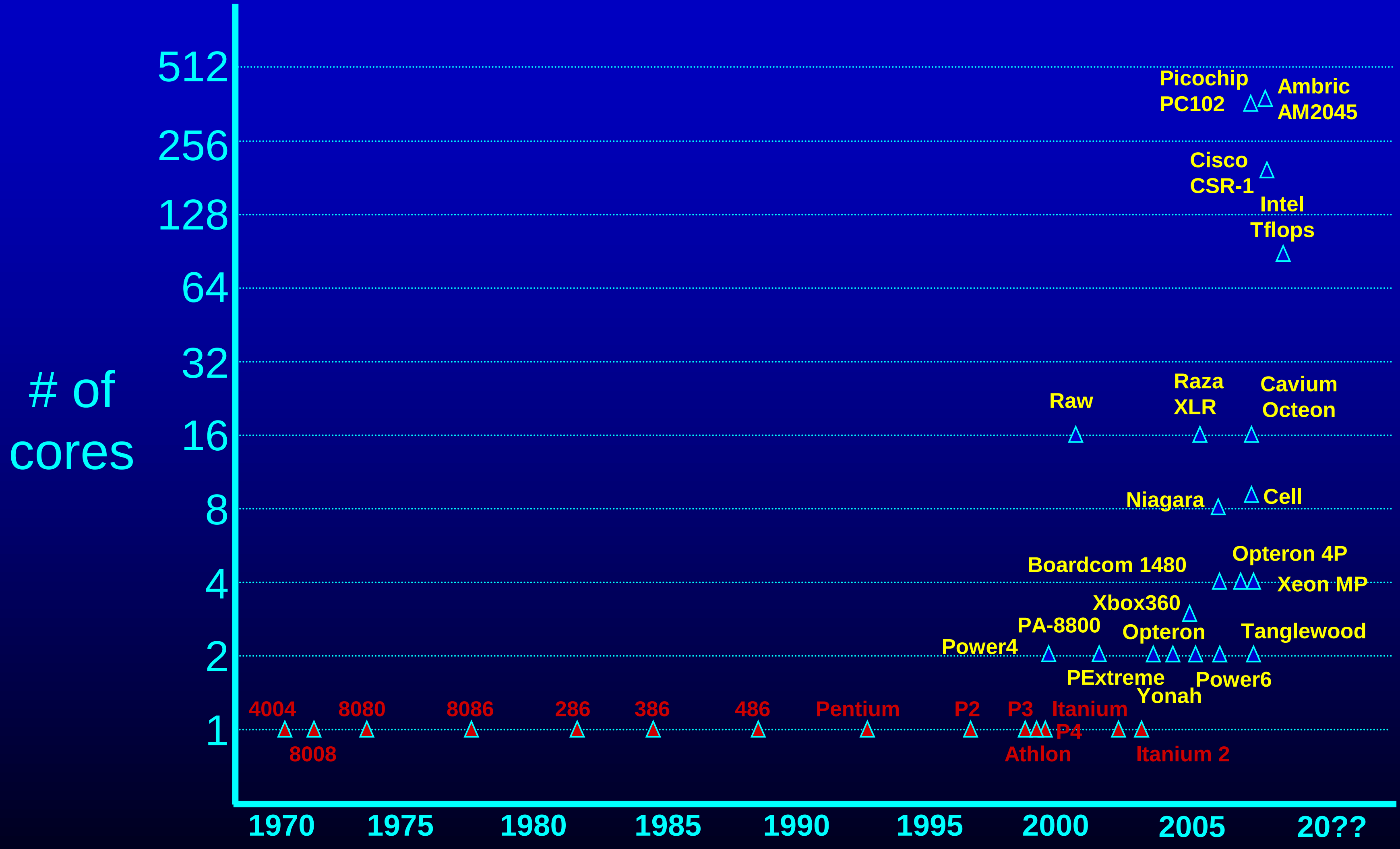
From David Patterson

Uniprocessor Performance (SPECint)



From David Patterson

Multicores Are Here!



Issues with Parallelism

- Amdhal's Law
  - Any computation can be analyzed in terms of a portion that must be executed sequentially, Ts, and a portion that can be executed in parallel, Tp. Then for n processors:
  - $T(n) = Ts + Tp/n$
  - $T(\infty) = Ts$, thus maximum speedup $(Ts + Tp)/Ts$
- Load Balancing
  - The work is distributed among processors so that ***all*** processors are kept busy when parallel task is executed.
- Granularity
  - The size of the parallel regions between synchronizations or the ratio of computation (useful work) to communication (overhead).

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Types of Parallelism

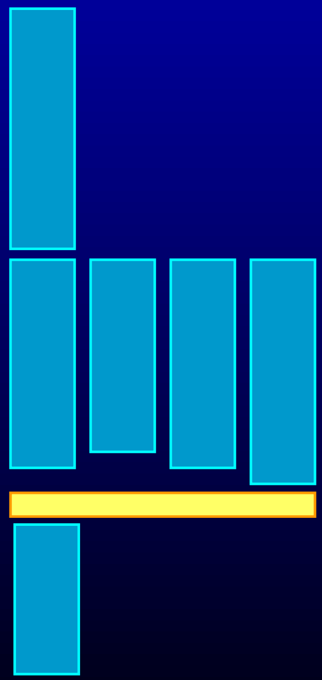
- Instruction Level Parallelism (ILP) → Scheduling and Hardware
- Task Level Parallelism (TLP) → Mainly by hand
- Loop Level Parallelism (LLP) or Data Parallelism → Hand or Compiler Generated
- Pipeline Parallelism → Hardware or Streaming
- Divide and Conquer Parallelism → Recursive functions

Why Loops?

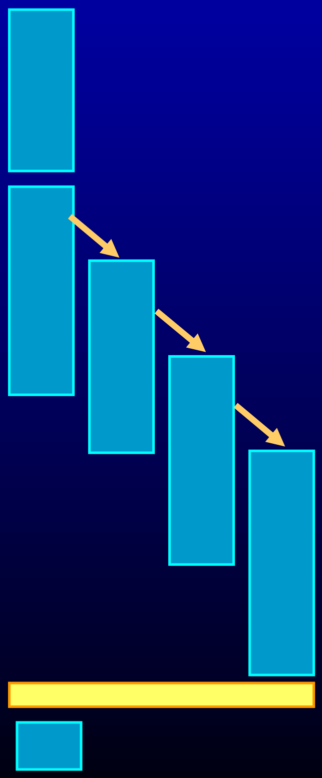
- 90% of the execution time in 10% of the code
  - Mostly in loops
- If parallel, can get good performance
  - Load balancing
- Relatively easy to analyze

Programmer Defined Parallel Loop

- FORALL
  - No "loop carried dependences"
  - Fully parallel



- FORACROSS
  - Some "loop carried dependences"



Parallel Execution

- Example

```
FORPAR I = 0 to N
  A[I] = A[I] + 1
```

- Block Distribution: Program gets mapped into

```
Iters = ceiling(N/NUMPROC);
FOR P = 0 to NUMPROC-1
  FOR I = P*Iters to MIN((P+1)*Iters, N)
   A[I] = A[I] + 1
```

- SPMD (Single Program, Multiple Data) Code

```
If(myPid == 0) {
  …
  Iters = ceiling(N/NUMPROC);
}
Barrier();
FOR I = myPid*Iters to MIN((myPid+1)*Iters, N)
  A[I] = A[I] + 1
Barrier();
```

Parallel Execution

- Example

```
FORPAR I = 0 to N
  A[I] = A[I] + 1
```

- Block Distribution: Program gets mapped into

```
Iters = ceiling(N/NUMPROC);
FOR P = 0 to NUMPROC-1
  FOR I = P*Iters to MIN((P+1)*Iters, N)
   A[I] = A[I] + 1
```

- Code fork a function

```
Iters = ceiling(N/NUMPROC);
FOR P = 0 to NUMPROC - 1 { ParallelExecute(func1, P); }
BARRIER(NUMPROC);
void func1(integer myPid)
{
  FOR I = myPid*Iters to MIN((myPid+1)*Iters, N)
   A[I] = A[I] + 1
}
```

Parallel Thread Basics

- Create separate threads
  - Create an OS thread
    - (hopefully) it will be run on a separate core
  - pthread_create(&thr, NULL, &entry_point, NULL)
  - Overhead in thread creation
    - Create a separate stack
    - Get the OS to allocate a thread
- Thread pool
  - Create all the threads (= num cores) at the beginning
  - Keep N-1 idling on a barrier, while sequential execution
  - Get them to run parallel code by each executing a function
  - Back to the barrier when parallel region is done

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- Why Parallelism
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- **Parallelizing Compilers**
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Parallelizing Compilers

- Finding FORALL Loops out of FOR loops

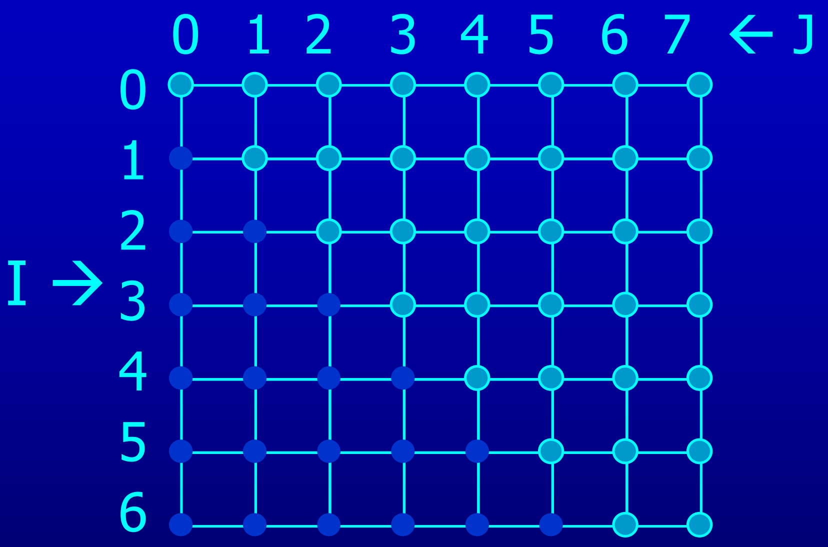
- Examples

```
FOR I = 0 to 5
  A[I] = A[I] + 1

FOR I = 0 to 5
  A[I] = A[I+6] + 1

For I = 0 to 5
  A[2*I] = A[2*I + 1] + 1
```

Iteration Space

- N deep loops → N-dimensional discrete iteration space
  - Normalized loops: assume step size = 1

```
FOR I = 0 to 6
   FOR J = I to 7
```

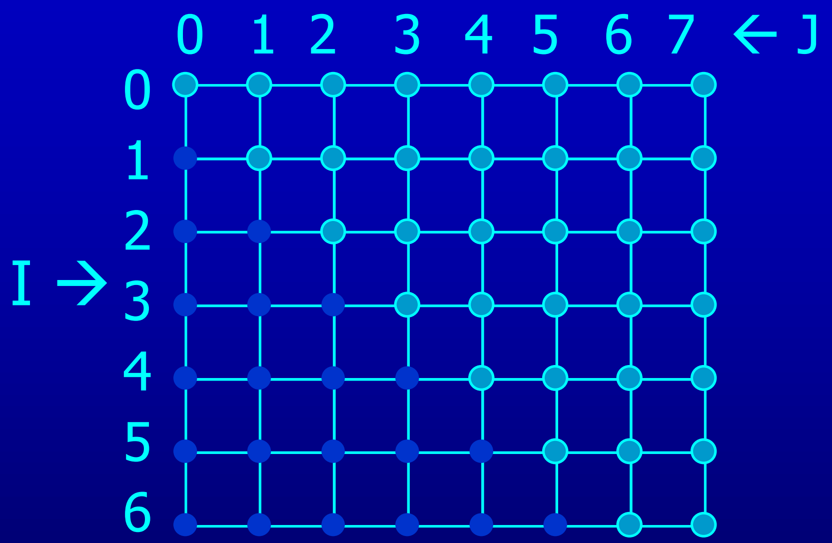


- Iterations are represented as coordinates in iteration space
  - $\bar{i} = [i_1, i_2, i_3, \ldots, i_n]$

Iteration Space

- N deep loops → N-dimensional discrete iteration space
  - Normalized loops: assume step size = 1

```
FOR I = 0 to 6
   FOR J = I to 7
```



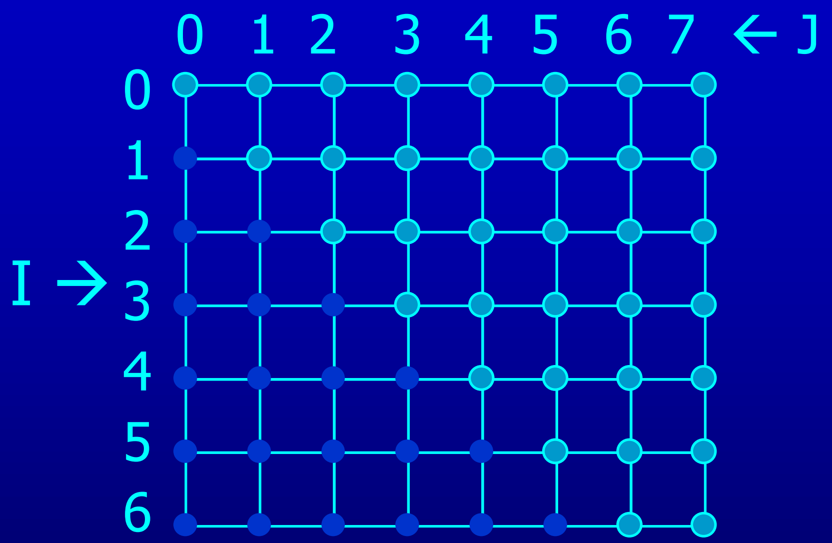
- Iterations are represented as coordinates in iteration space
- Sequential execution order of iterations → Lexicographic order

[0,0], [0,1], [0,2], …, [0,6], [0,7],
[1,1], [1,2], …, [1,6], [1,7],
[2,2], …, [2,6], [2,7],
………
[6,6], [6,7],

Iteration Space

- N deep loops → N-dimensional discrete iteration space
  - Normalized loops: assume step size = 1

```
FOR I = 0 to 6
   FOR J = I to 7
```

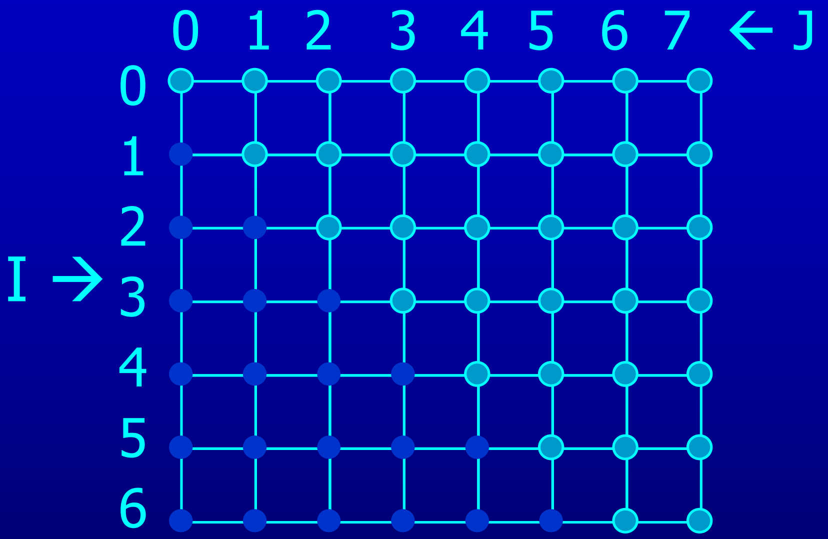


- Iterations are represented as coordinates in iteration space
- Sequential execution order of iterations → Lexicographic order
- Iteration $\bar{i}$ is lexicograpically less than $\bar{j}$, $\bar{i} < \bar{j}$ iff
  there exists c s.t. $i_1 = j_1$, $i_2 = j_2$,… $i_{c-1} = j_{c-1}$ and $i_c < j_c$

Iteration Space

- N deep loops → N-dimensional discrete iteration space
  - Normalized loops: assume step size = 1

```
FOR I = 0 to 6
   FOR J = I to 7
```

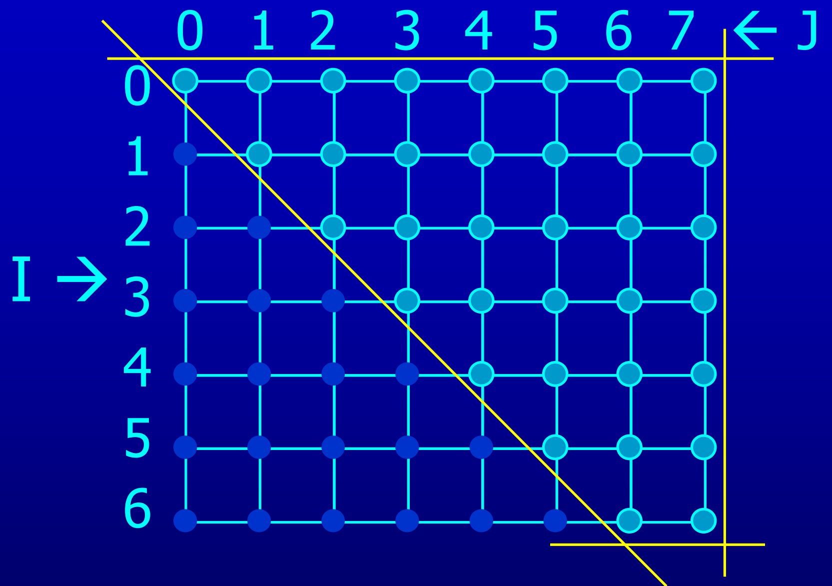


- An affine loop nest
  - Loop bounds are integer linear functions of constants, loop constant variables and outer loop indexes
  - Array accesses are integer linear functions of constants, loop constant variables and loop indexes

Iteration Space

- N deep loops → N-dimensional discrete iteration space
  - Normalized loops: assume step size = 1

```
FOR I = 0 to 6
   FOR J = I to 7
```



- Affine loop nest → Iteration space as a set of linear inequalities

$0 \le I$

$I \le 6$

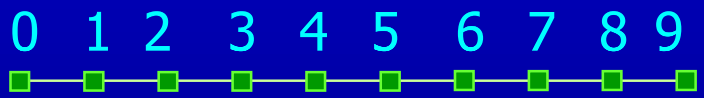
$I \le J$

$J \le 7$

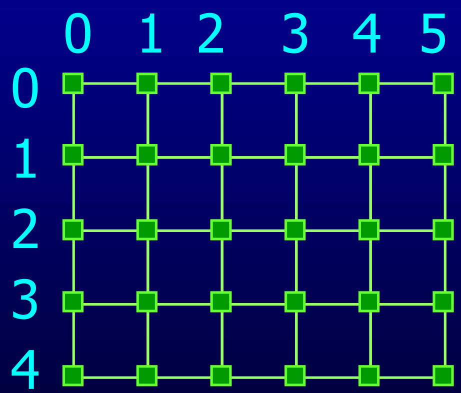
Data Space

- M dimensional arrays → M-dimensional discrete cartesian space
  - a hypercube

`Integer A(10)`



`Float B(5, 6)`



Dependences

- True dependence

```
a  =
   =  a
```

- Anti dependence

```
   =  a
a  =
```

- Output dependence

```
a  =
a  =
```

- Definition:
  Data dependence exists for a dynamic instance i and j iff
  - either i or j is a write operation
  - i and j refer to the same variable
  - i executes before j

- How about array accesses within loops?

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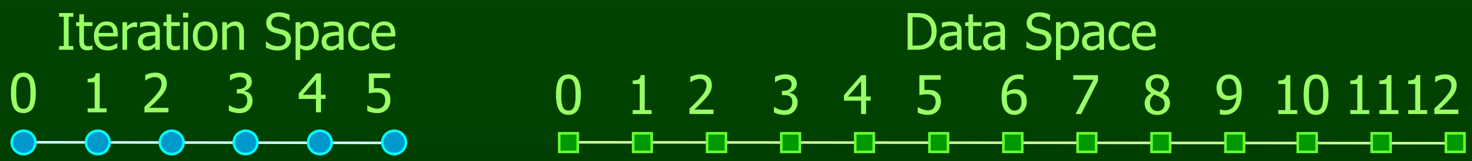
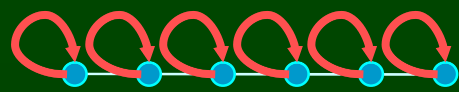
Array Accesses in a loop

```
FOR I = 0 to 5
  A[I] = A[I] + 1
```



Array Accesses in a loop



```
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  A[I] = A[I] + 1
```
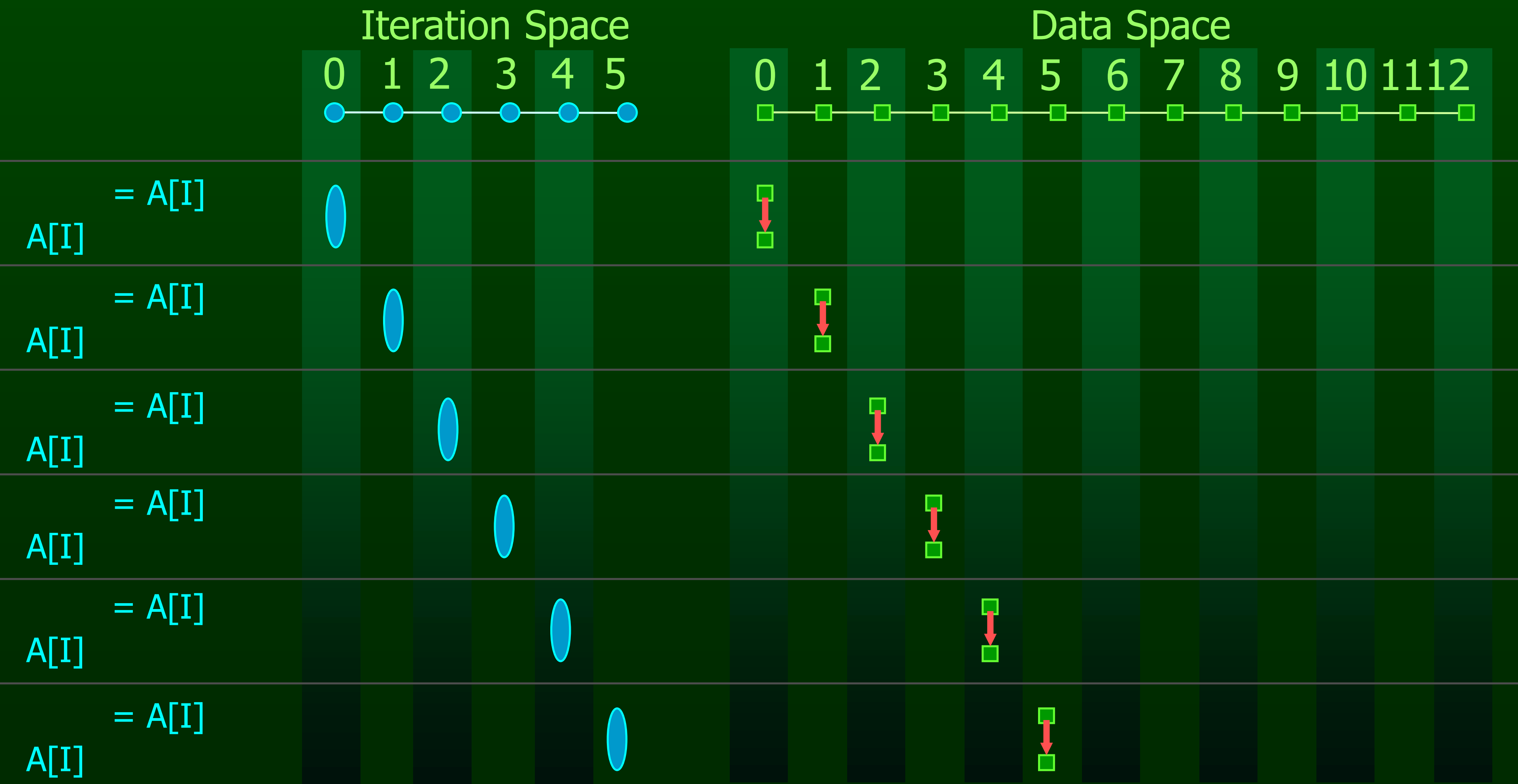
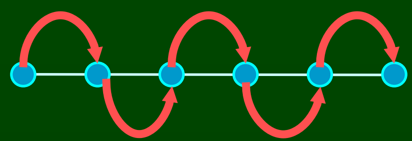
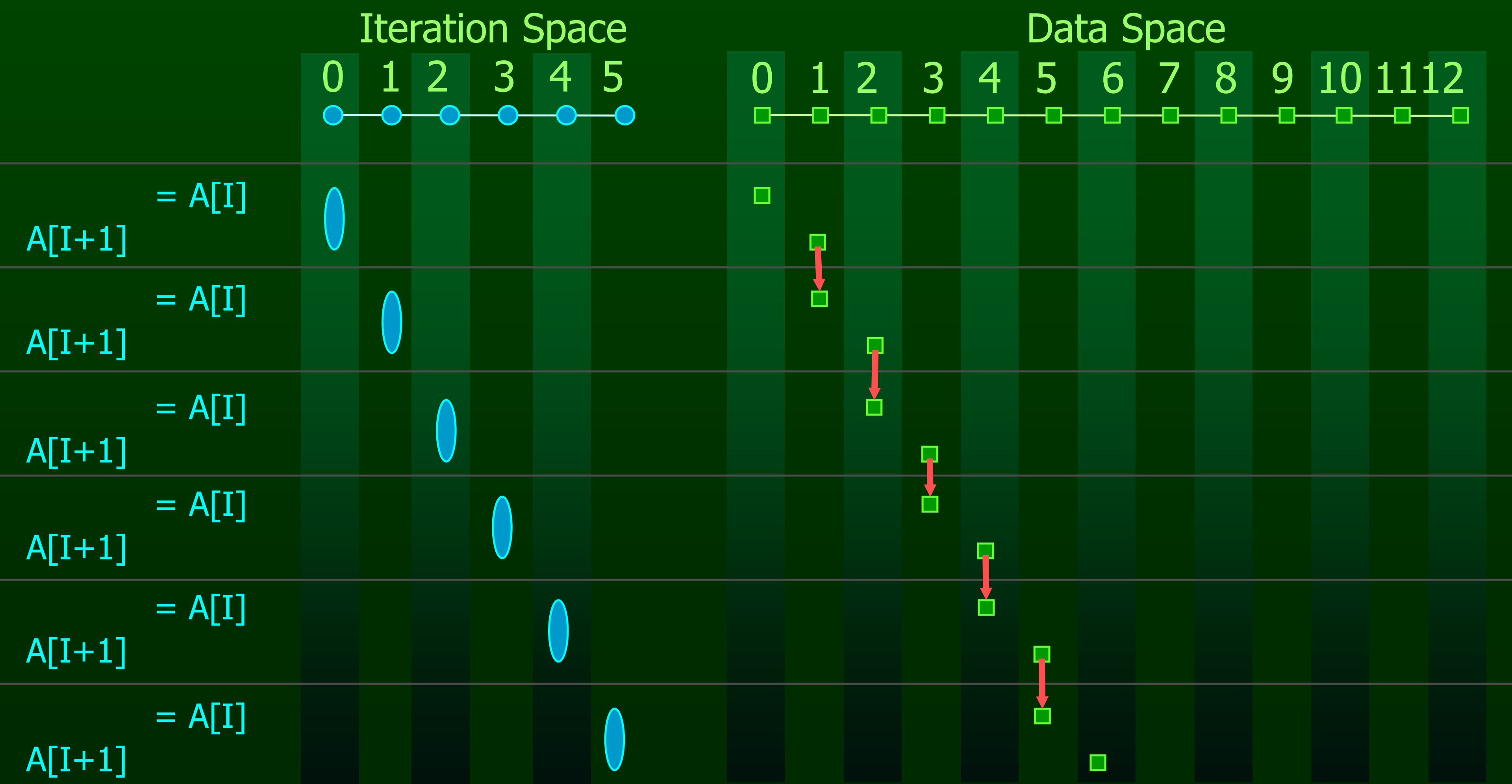



Array Accesses in a loop

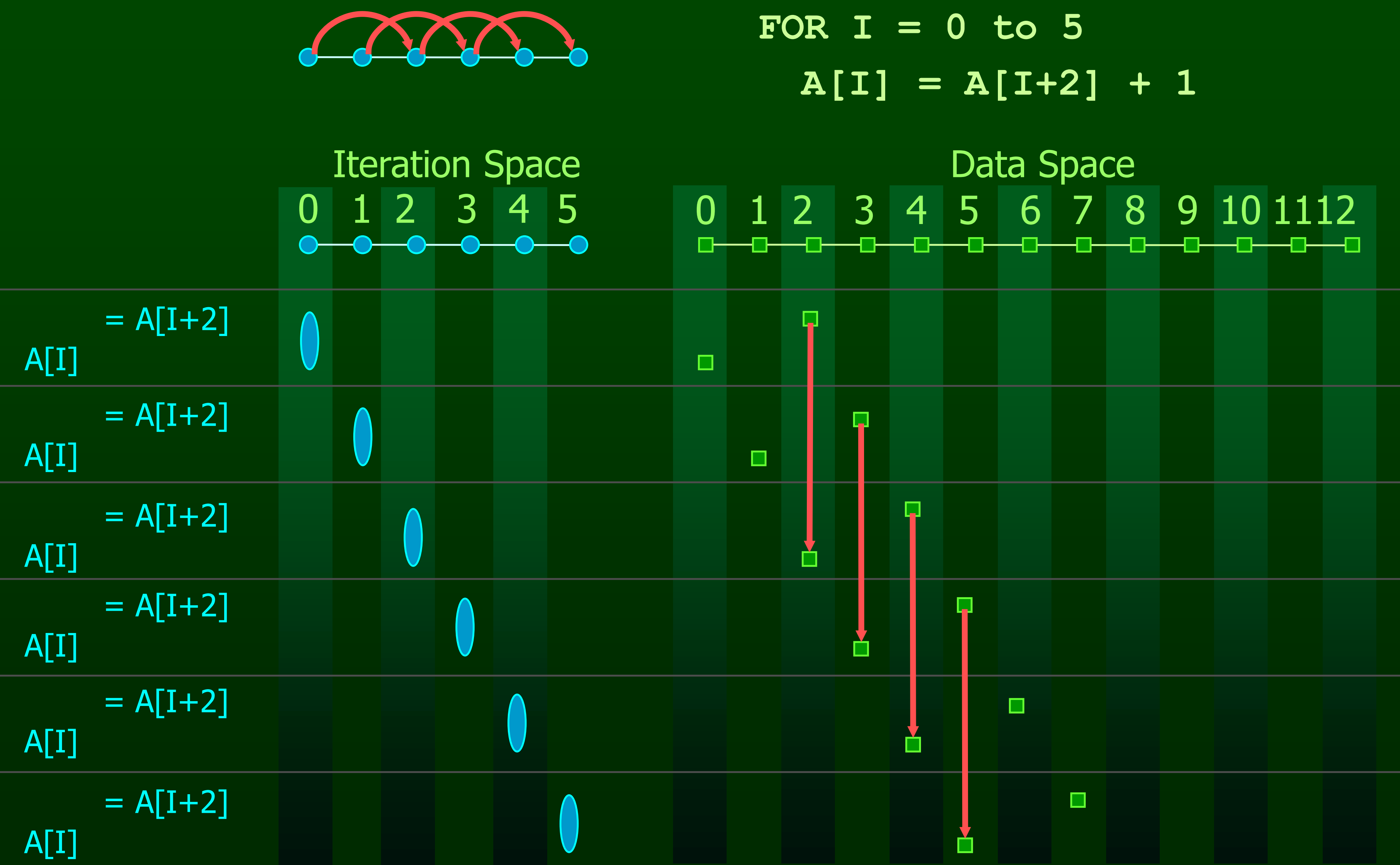


```
FOR I = 0 to 5
  A[I+1] = A[I] + 1
```



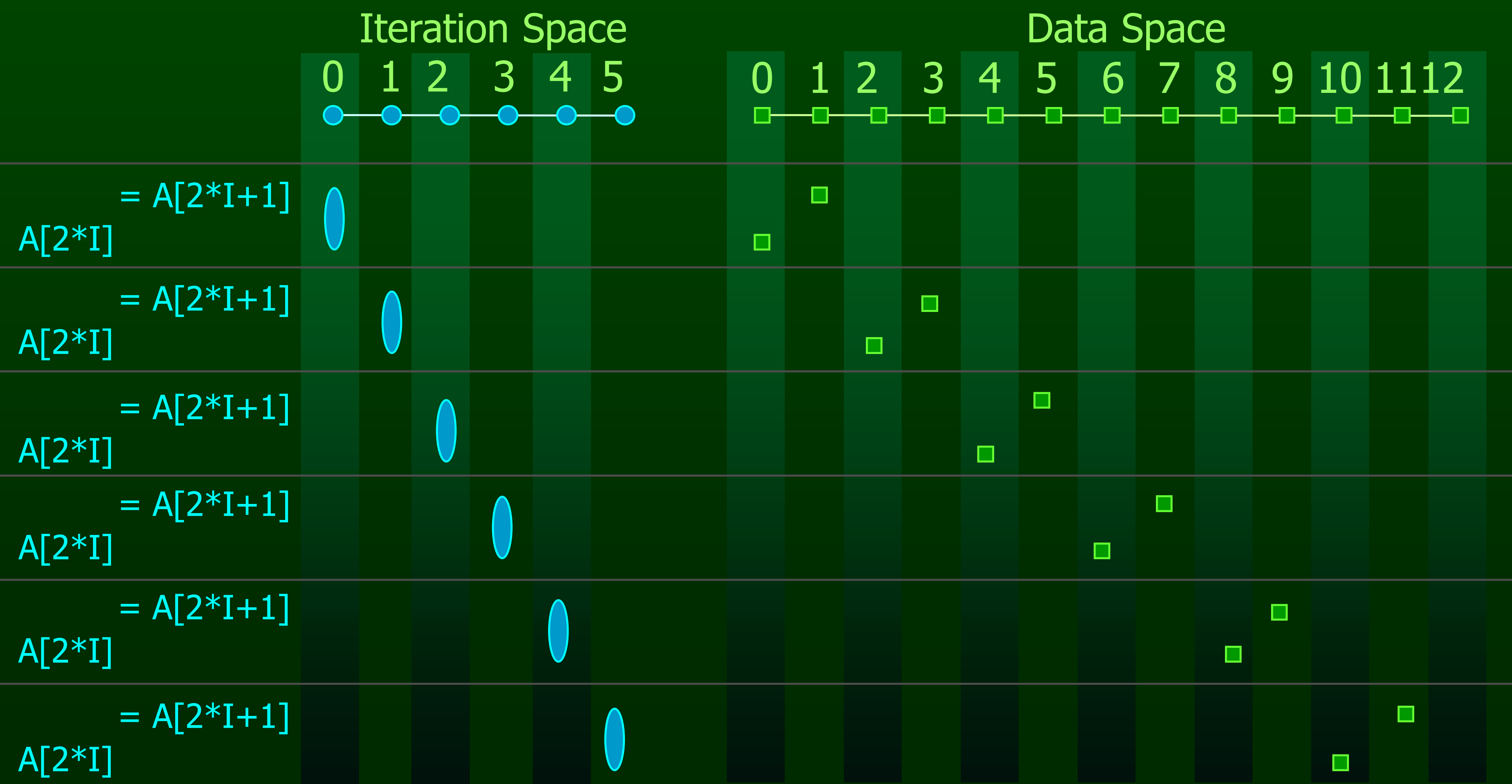
Array Accesses in a loop

```
FOR I = 0 to 5
  A[I] = A[I+2] + 1
```



Array Accesses in a loop

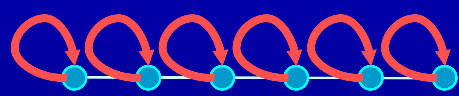
```
FOR I = 0 to 5
  A[2*I] = A[2*I+1] + 1
```



Distance Vectors

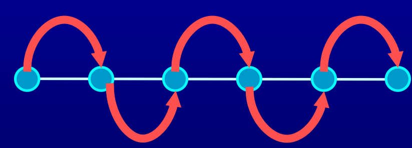
- A loop has a distance d if there exist a data dependence from iteration i to j and d = j-i

$$dv = [0]$$



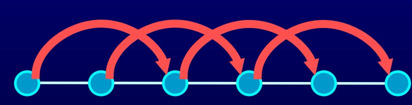
```
FOR I = 0 to 5
  A[I] = A[I] + 1
```

$$dv = [1]$$



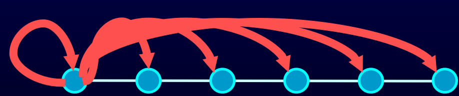
```
FOR I = 0 to 5
  A[I+1] = A[I] + 1
```

$$dv = [2]$$



```
FOR I = 0 to 5
  A[I] = A[I+2] + 1
```

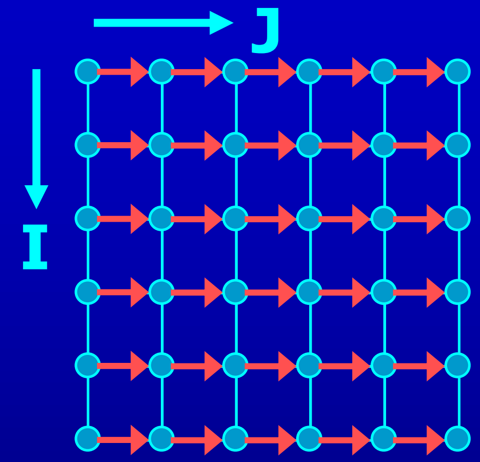
$$dv = [1], \quad [2] \quad \ldots \quad = \quad [*]$$



```
FOR I = 0 to 5
  A[I] = A[0] + 1
```

Multi-Dimensional Dependence

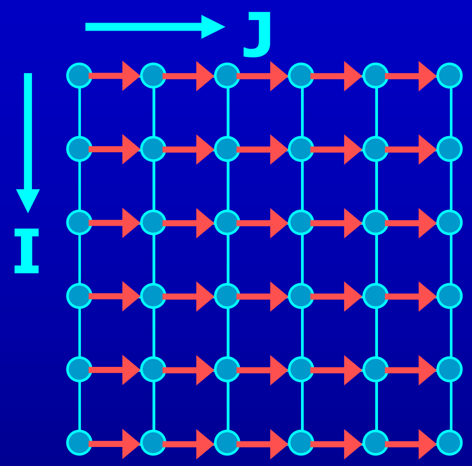
```
FOR I = 1 to n
  FOR J = 1 to n
   A[I, J] = A[I, J-1] + 1
```



$$dv = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

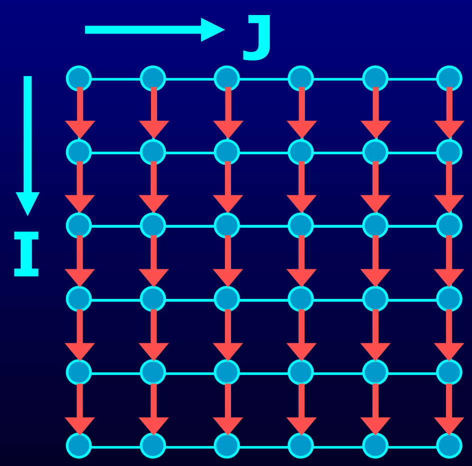
Multi-Dimensional Dependence

```
FOR I = 1 to n
  FOR J = 1 to n
   A[I, J] = A[I, J-1] + 1
```



$$dv = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

```
FOR I = 1 to n
  FOR J = 1 to n
   A[I, J] = A[I+1, J] + 1
```



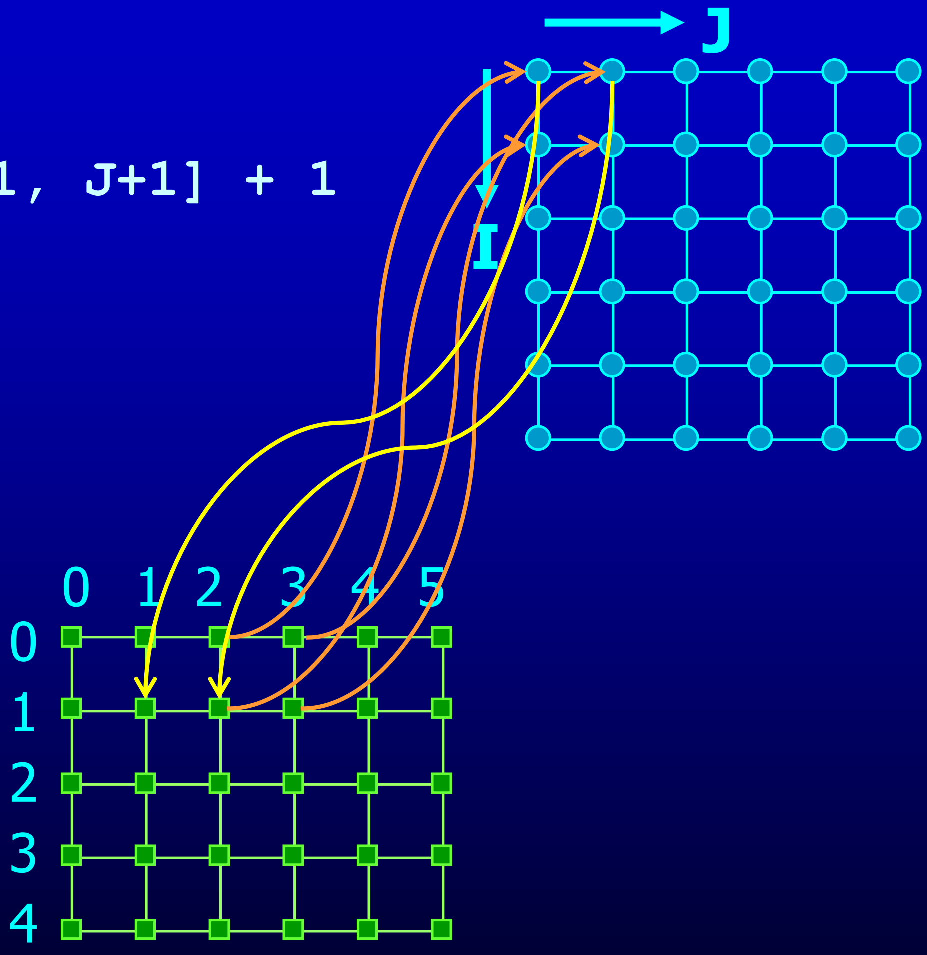
$$dv = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Outline

- Dependence Analysis
- Increasing Parallelization Opportunities

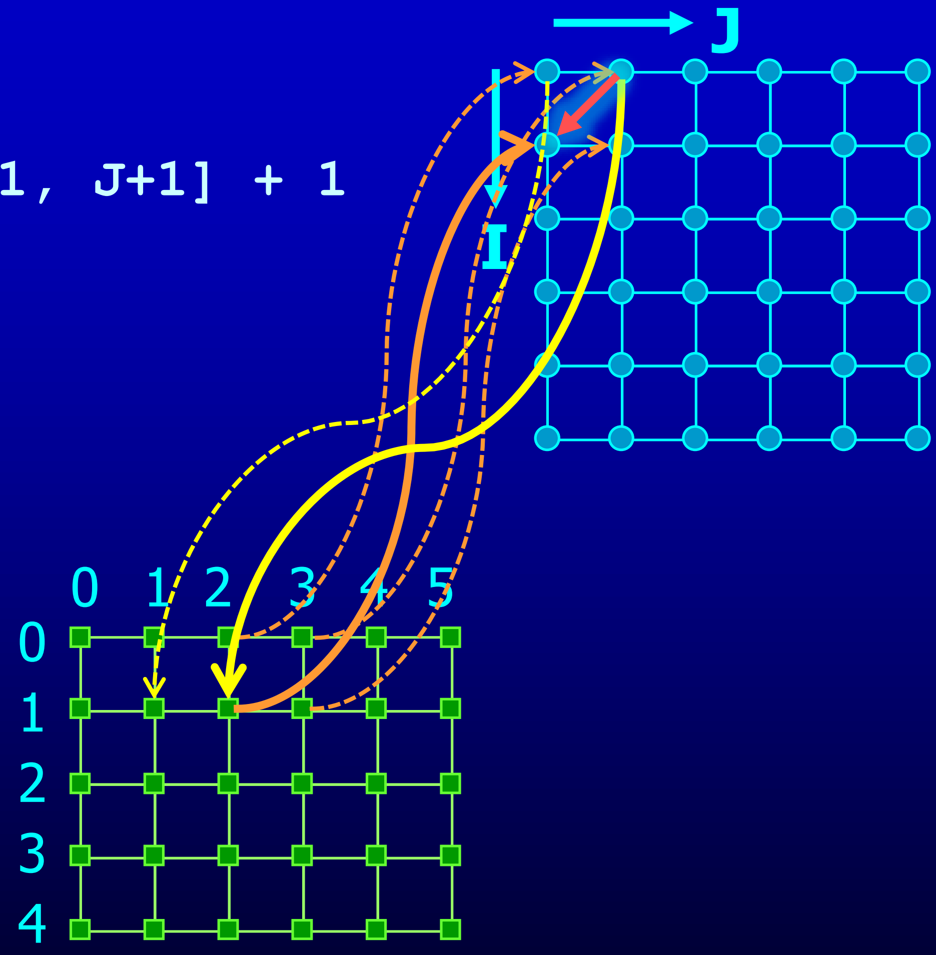
What is the Dependence?

```
FOR I = 1 to n
  FOR J = 1 to n
   A[I, J] = A[I-1, J+1] + 1
```



What is the Dependence?

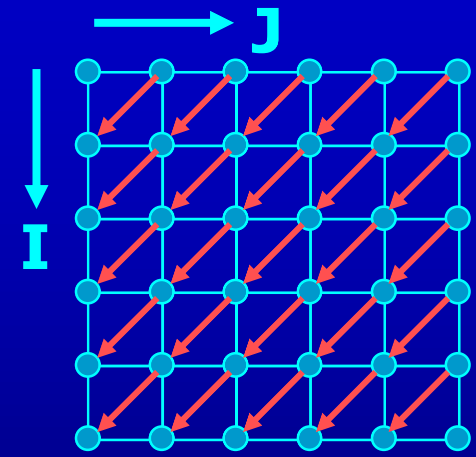
```
FOR I = 1 to n
  FOR J = 1 to n
   A[I, J] = A[I-1, J+1] + 1
```



What is the Dependence?

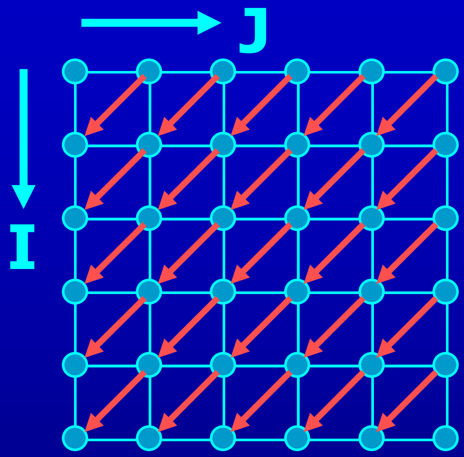
```
FOR I = 1 to n
  FOR J = 1 to n
   A[I, J] = A[I-1, J+1] + 1
```

$$dv = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

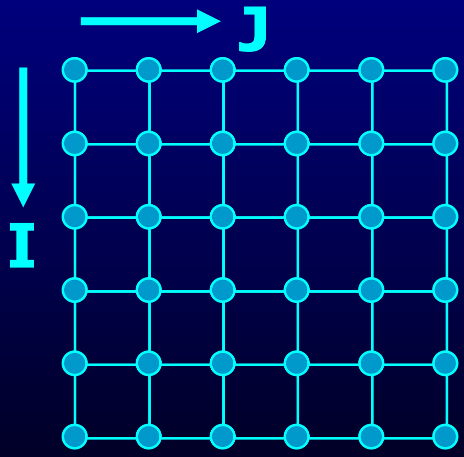


What is the Dependence?

```
FOR I = 1 to n
  FOR J = 1 to n
   A[I, J] = A[I-1, J+1] + 1
```



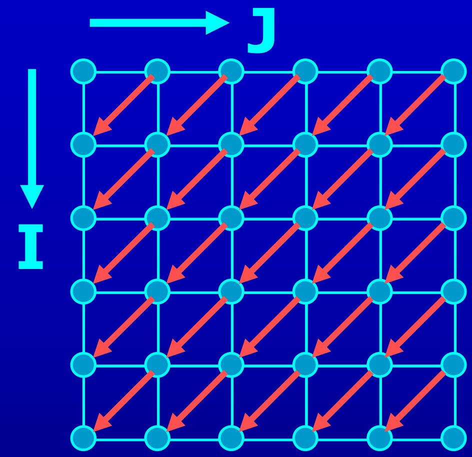

```
FOR I = 1 to n
  FOR J = 1 to n
   B[I] = B[I-1] + 1
```



What is the Dependence?

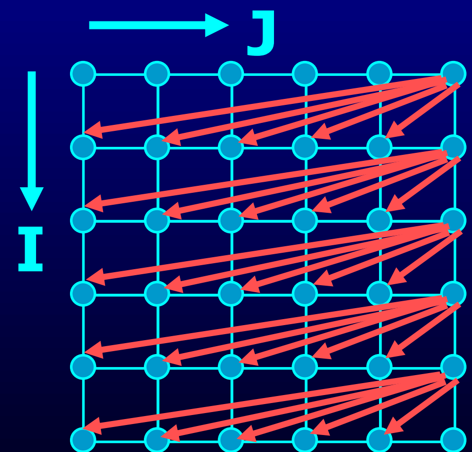
```
FOR I = 1 to n
  FOR J = 1 to n
   A[I, J] = A[I-1, J+1] + 1
```

$$dv = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



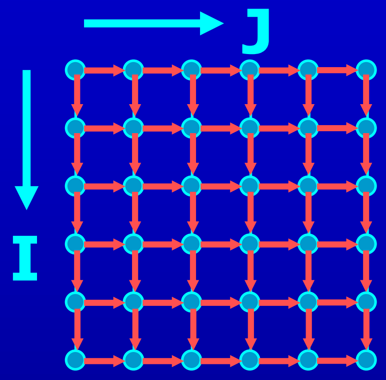

```
FOR I = 1 to n
  FOR J = 1 to n
   B[I] = B[I-1] + 1
```

$$dv = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \ldots = \begin{bmatrix} 1 \\ * \end{bmatrix}$$



What is the Dependence?

```
FOR i = 1 to N-1
  FOR j = 1 to N-1
   A[i,j] = A[i,j-1] + A[i-1,j];
```



$$dv = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Recognizing FORALL Loops

- Find data dependences in loop
  - For every pair of array acceses to the same array

    If the first access has at least one dynamic instance (an iteration) in which it refers to a location in the array that the second access also refers to in at least one of the later dynamic instances (iterations).

    Then there is a data dependence between the statements
  - (Note that same array can refer to itself – output dependences)
- Definition
  - Loop-carried dependence:
    dependence that crosses a loop boundary
- If there are no loop carried dependences → parallelizable

Data Dependence Analysis

- I: Distance Vector method
- II: Integer Programming

Distance Vector Method

- The i^{th} loop is parallelizable for all dependence $d = [d_1,\ldots,d_i,..d_n]$
  either

  one of $d_1,\ldots,d_{i-1}$ is > 0

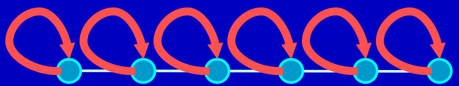
  or

  all $d_1,\ldots,d_i = 0$

Is the Loop Parallelizable?

$dv = [0]$

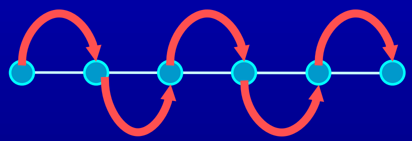
Yes



```
FOR I = 0 to 5
  A[I] = A[I] + 1
```

$dv = [1]$

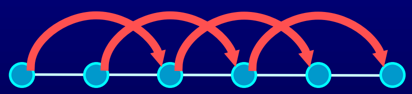
No



```
FOR I = 0 to 5
  A[I+1] = A[I] + 1
```

$dv = [2]$

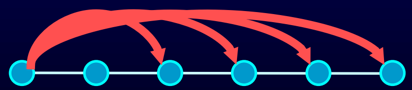
No



```
FOR I = 0 to 5
  A[I] = A[I+2] + 1
```

$dv = [*]$

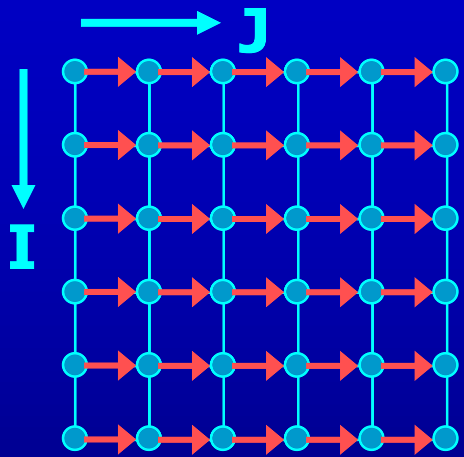
No



```
FOR I = 0 to 5
  A[I] = A[0] + 1
```

Are the Loops Parallelizable?

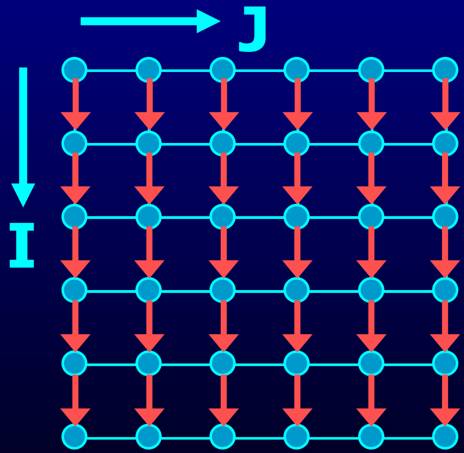
```
FOR I = 1 to n
  FOR J = 1 to n
   A[I, J] = A[I, J-1] + 1
```



$$dv = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Yes
No

```
FOR I = 1 to n
  FOR J = 1 to n
   A[I, J] = A[I+1, J] + 1
```



$$dv = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

No
Yes

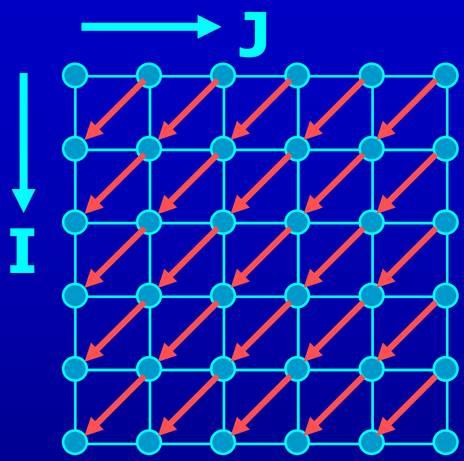
Are the Loops Parallelizable?

```
FOR I = 1 to n
  FOR J = 1 to n
   A[I, J] = A[I-1, J+1] + 1
```

$$dv = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

No

Yes

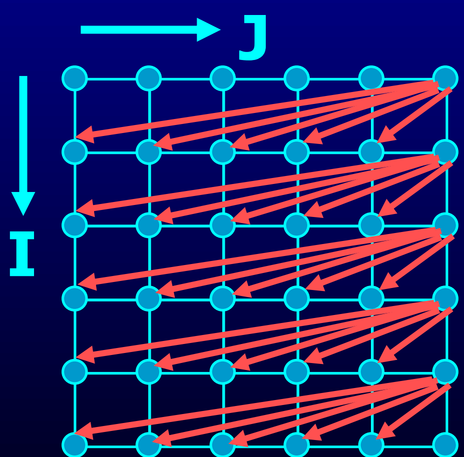



```
FOR I = 1 to n
  FOR J = 1 to n
   B[I] = B[I-1] + 1
```

$$dv = \begin{bmatrix} 1 \\ * \end{bmatrix}$$

No

Yes



Integer Programming Method

- Example

```
FOR I = 0 to 5
  A[I+1] = A[I] + 1
```

- Is there a loop-carried dependence between A[I+1] and A[I]
  - Are there two distinct iterations i_w and i_r such that $A[i_w+1]$ is the same location as $A[i_r]$
  - $\exists$ integers i_w, i_r $\quad 0 \leq i_w, i_r \leq 5 \quad i_w \neq i_r \quad i_w + 1 = i_r$
- Is there a dependence between A[I+1] and A[I+1]
  - Are there two distinct iterations i_1 and i_2 such that $A[i_1+1]$ is the same location as $A[i_2+1]$
  - $\exists$ integers i_1, i_2 $\quad 0 \leq i_1, i_2 \leq 5 \quad i_1 \neq i_2 \quad i_1 + 1 = i_2 + 1$

Integer Programming Method

```
FOR I = 0 to 5
  A[I+1] = A[I] + 1
```

- Formulation
  - $\exists$ an integer vector $\bar{\imath}$ such that $\hat{A}\bar{\imath} \le \bar{b}$ where $\hat{A}$ is an integer matrix and $\bar{b}$ is an integer vector

Iteration Space

```
FOR I = 0 to 5
  A[I+1] = A[I] + 1
```

- N deep loops → n-dimensional discrete cartesian space

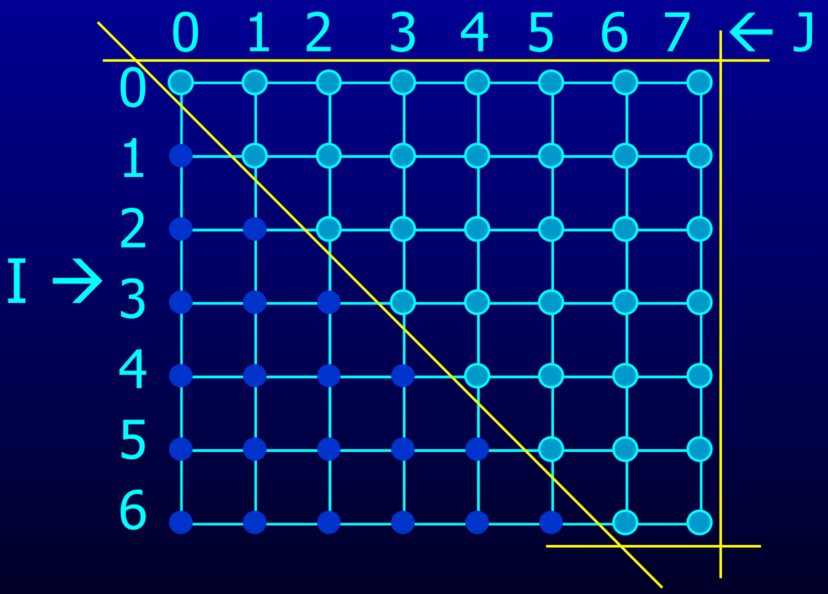
- Affine loop nest → Iteration space as a set of linear inequalities

  $0 \leq I$

  $I \leq 6$

  $I \leq J$

  $J \leq 7$



Integer Programming Method

```
FOR I = 0 to 5
  A[I+1] = A[I] + 1
```

- Formulation
  - $\exists$ an integer vector $\bar{\imath}$ such that $\hat{A}\bar{\imath} \le \bar{b}$ where $\hat{A}$ is an integer matrix and $\bar{b}$ is an integer vector
- Our problem formulation for A[i] and A[i+1]
  - $\exists$ integers i_w, i_r $0 \le i_w, i_r \le 5$ $i_w \ne i_r$ $i_w + 1 = i_r$
  - $i_w \ne i_r$ is not an affine function
    - divide into 2 problems
    - Problem 1 with $i_w < i_r$ and problem 2 with $i_r < i_w$
    - If either problem has a solution → there exists a dependence
  - How about $i_w + 1 = i_r$
    - Add two inequalities to single problem $i_w + 1 \le i_r$, and $i_r \le i_w + 1$

Integer Programming Formulation

```
FOR I = 0 to 5
  A[I+1] = A[I] + 1
```

- Problem 1

  $0 \leq i_w$

  $i_w \leq 5$

  $0 \leq i_r$

  $i_r \leq 5$

  $i_w < i_r$

  $i_w + 1 \leq i_r$

  $i_r \leq i_w + 1$

Integer Programming Formulation

```
FOR I = 0 to 5
  A[I+1] = A[I] + 1
```

- Problem 1

$0 \leq i_w$	$\rightarrow$	$-i_w \leq 0$
$i_w \leq 5$	$\rightarrow$	$i_w \leq 5$
$0 \leq i_r$	$\rightarrow$	$-i_r \leq 0$
$i_r \leq 5$	$\rightarrow$	$i_r \leq 5$
$i_w < i_r$	$\rightarrow$	$i_w - i_r \leq -1$
$i_w + 1 \leq i_r$	$\rightarrow$	$i_w - i_r \leq -1$
$i_r \leq i_w + 1$	$\rightarrow$	$-i_w + i_r \leq 1$

Integer Programming Formulation

- Problem 1

$$
\begin{array}{lcl}
0 \le i_w & \rightarrow & -i_w \le 0 \\
i_w \le 5 & \rightarrow & i_w \le 5 \\
0 \le i_r & \rightarrow & -i_r \le 0 \\
i_r \le 5 & \rightarrow & i_r \le 5 \\
i_w < i_r & \rightarrow & i_w - i_r \le -1 \\
i_w + 1 \le i_r & \rightarrow & i_w - i_r \le -1 \\
i_r \le i_w + 1 & \rightarrow & -i_w + i_r \le 1
\end{array}
\qquad
\hat{A} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}
\qquad
\bar{b} = \begin{pmatrix} 0 \\ 5 \\ 0 \\ 5 \\ -1 \\ -1 \\ 1 \end{pmatrix}
$$

- and problem 2 with $i_r < i_w$

Generalization

- An affine loop nest

```
FOR i_1 = f_11(c_1…c_k) to I_u1(c_1…c_k)
  FOR i_2 = f_12(i_1,c_1…c_k) to I_u2(i_1,c_1…c_k)
    ……
      FOR i_n = f_1n(i_1…i_n-1,c_1…c_k) to I_un(i_1…i_n-1,c_1…c_k)
          A[f_a1(i_1…i_n,c_1…c_k), f_a2(i_1…i_n,c_1…c_k),…,f_am(i_1…i_n,c_1…c_k)]
```

- Solve 2*n problems of the form
  - $i_1 = j_1,\ i_2 = j_2, \ldots\ i_{n-1} = j_{n-1},\ i_n < j_n$
  - $i_1 = j_1,\ i_2 = j_2, \ldots\ i_{n-1} = j_{n-1},\ j_n < i_n$
  - $i_1 = j_1,\ i_2 = j_2, \ldots\ i_{n-1} < j_{n-1}$
  - $i_1 = j_1,\ i_2 = j_2, \ldots\ j_{n-1} < i_{n-1}$

  ………………

  - $i_1 = j_1,\ i_2 < j_2$
  - $i_1 = j_1,\ j_2 < i_2$
  - $i_1 < j_1$
  - $j_1 < i_1$

Outline

- Why Parallelism
- Parallel Execution
- Parallelizing Compilers
- Dependence Analysis
- **Increasing Parallelization Opportunities**

Increasing Parallelization Opportunities

- Scalar Privatization
- Reduction Recognition
- Induction Variable Identification
- Array Privatization
- Loop Transformations
- Granularity of Parallelism
- Interprocedural Parallelization

Scalar Privatization

- Example

```
FOR i = 1 to n
  X = A[i] * 3;
  B[i] = X;
```

- Is there a loop carried dependence?
- What is the type of dependence?

Privatization

- Analysis:
  - Any anti- and output- loop-carried dependences

- Eliminate by assigning in local context

```
FOR i = 1 to n
  integer Xtmp;
  Xtmp = A[i] * 3;
  B[i] = Xtmp;
```

- Eliminate by expanding into an array

```
FOR i = 1 to n
  Xtmp[i] = A[i] * 3;
  B[i] = Xtmp[i];
```

Privatization

- Need a final assignment to maintain the correct value after the loop nest
- Eliminate by assigning in local context

```
FOR i = 1 to n
  integer Xtmp;
  Xtmp = A[i] * 3;
  B[i] = Xtmp;
  if(i == n) X = Xtmp
```

- Eliminate by expanding into an array

```
FOR i = 1 to n
  Xtmp[i] = A[i] * 3;
  B[i] = Xtmp[i];
X = Xtmp[n];
```

Another Example

- How about loop-carried true dependences?
- Example

```
FOR i = 1 to n
  X = X + A[i];
```

- Is this loop parallelizable?

Reduction Recognition

- Reduction Analysis:
  - Only associative operations
  - The result is never used within the loop

- Transformation

```
Integer Xtmp[NUMPROC];
Barrier();
FOR i = myPid*Iters to MIN((myPid+1)*Iters, n)
   Xtmp[myPid] = Xtmp[myPid] + A[i];
Barrier();
If(myPid == 0) {
  FOR p = 0 to NUMPROC-1
   X = X + Xtmp[p];
  …
```

Induction Variables

- Example

```
FOR i = 0 to N
  A[i] = 2^i;
```

- After strength reduction

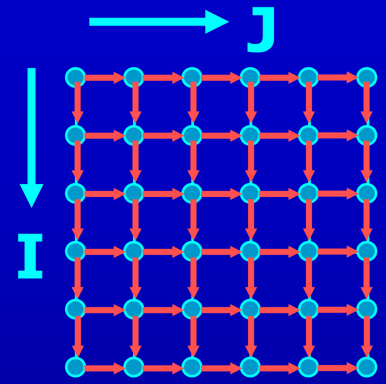
```
t = 1
FOR i = 0 to N
  A[i] = t;
  t = t*2;
```

- What happened to loop carried dependences?
- Need to do opposite of this!
  - Perform induction variable analysis
  - Rewrite IVs as a function of the loop variable

Array Privatization

- Similar to scalar privatization
- However, analysis is more complex
  - Array Data Dependence Analysis:
    Checks if two iterations access the same location
  - Array Data Flow Analysis:
    Checks if two iterations access the same value
- Transformations
  - Similar to scalar privatization
  - Private copy for each processor or expand with an additional dimension

Loop Transformations



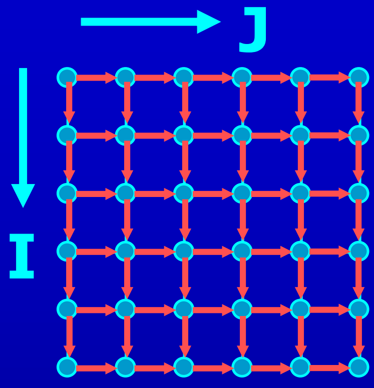
- A loop may not be parallel as is
- Example

```
FOR i = 1 to N-1
  FOR j = 1 to N-1
   A[i,j] = A[i,j-1] + A[i-1,j];
```

Loop Transformations

- A loop may not be parallel as is
- Example

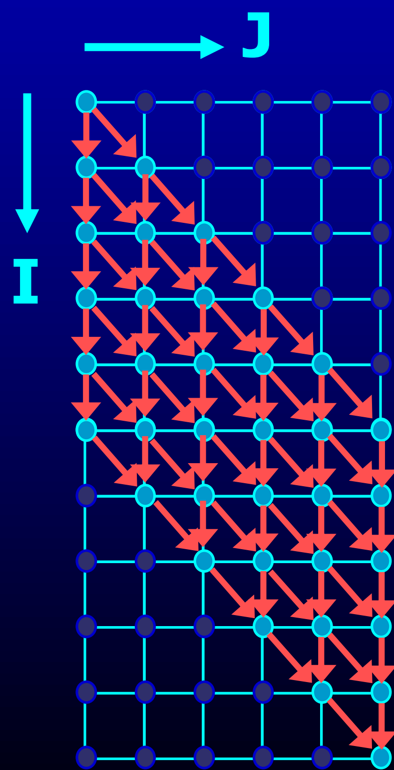
```
FOR i = 1 to N-1
  FOR j = 1 to N-1
   A[i,j] = A[i,j-1] + A[i-1,j];
```



- After loop Skewing

$$\begin{bmatrix} i_{new} \\ j_{new} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_{old} \\ j_{old} \end{bmatrix}$$

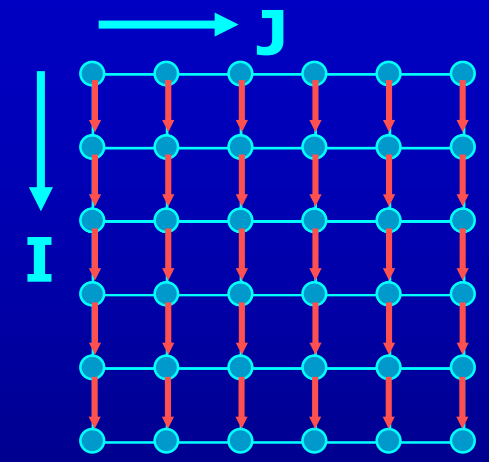
```
FOR i = 1 to 2*N-3
  FORPAR j = max(1,i-N+2) to min(i, N-1)
   A[i-j+1,j] = A[i-j+1,j-1] + A[i-j,j];
```



Granularity of Parallelism

- Example

```
FOR i = 1 to N-1
   FOR j = 1 to N-1
    A[i,j] = A[i,j] + A[i-1,j];
```



- Gets transformed into

```
FOR i = 1 to N-1
   Barrier();
   FOR j = 1+ myPid*Iters to MIN((myPid+1)*Iters, n-1)
     A[i,j] = A[i,j] + A[i-1,j];
   Barrier();
```

- Inner loop parallelism can be expensive
  - Startup and teardown overhead of parallel regions
  - Lot of synchronization
  - Can even lead to slowdowns

Granularity of Parallelism

- Inner loop parallelism can be expensive

- Solutions
  - Don't parallelize if the amount of work within the loop is too small

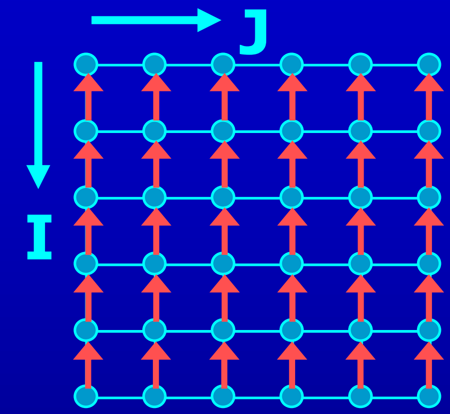
  or

  - Transform into outer-loop parallelism

Outer Loop Parallelism

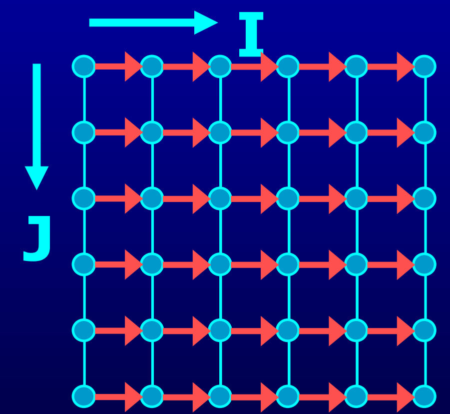
- Example

```
FOR i = 1 to N-1
   FOR j = 1 to N-1
      A[i,j] = A[i,j] + A[i-1,j];
```



- After Loop Transpose

```
FOR j = 1 to N-1
   FOR i = 1 to N-1
      A[i,j] = A[i,j] + A[i-1,j];
```



- Get mapped into

```
Barrier();
FOR j = 1+ myPid*Iters to MIN((myPid+1)*Iters, n-1)
   FOR i = 1 to N-1
      A[i,j] = A[i,j] + A[i-1,j];
Barrier();
```

Unimodular Transformations

- Interchange, reverse and skew
- Use a matrix transformation

  $$I_{new} = A\ I_{old}$$

- Interchange $\begin{bmatrix} i_{new} \\ j_{new} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i_{old} \\ j_{old} \end{bmatrix}$
- Reverse $\begin{bmatrix} i_{new} \\ j_{new} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_{old} \\ j_{old} \end{bmatrix}$
- Skew $\begin{bmatrix} i_{new} \\ j_{new} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_{old} \\ j_{old} \end{bmatrix}$

Legality of Transformations

- Unimodular transformation with matrix A is valid iff.
  For all dependence vectors v

  the first non-zero in Av is positive
- Example

```
FOR i = 1 to N-1
   FOR j = 1 to N-1
     A[i,j] = A[i,j] + A[i-1,j];
```

$$dv = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Interchange $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ✓
- Reverse $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ ✗
- Skew $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ ✓✓

Interprocedural Parallelization

- Function calls will make a loop unparallelizatble
  - Reduction of available parallelism
  - A lot of inner-loop parallelism
- Solutions
  - Interprocedural Analysis
  - Inlining

Interprocedural Parallelization

- Issues
  - Same function reused many times
  - Analyze a function on each trace → Possibly exponential
  - Analyze a function once → unrealizable path problem

- Interprocedural Analysis
  - Need to update all the analysis
  - Complex analysis
  - Can be expensive

- Inlining
  - Works with existing analysis
  - Large code bloat → can be very expensive

```
HashSet h;
for i = 1 to n
    int v = compute(i);
    h.insert(i);
```

Are iterations independent?

Can you still execute the loop in parallel?

Do all parallel executions give same result?

Summary

- Multicores are here
  - Need parallelism to keep the performance gains
  - Programmer defined or compiler extracted parallelism
- Automatic parallelization of loops with arrays
  - Requires Data Dependence Analysis
  - Iteration space & data space abstraction
  - An integer programming problem
- Many optimizations that'll increase parallelism